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Non-integrability of the generalized spring-pendulum problem

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Abstract
We investigate a generalization of the three-dimensional spring-pendulum system. The problem depends on two real parameters (k, a), where k is the Young modulus of the spring and a describes the nonlinearity of elastic forces. We show that this system is not integrable when $k \neq -a$. We carefully investigated the case $k = -a$ when the necessary condition for integrability given by the Morales-Ruiz–Ramis theory is satisfied. We discuss an application of the higher order variational equations for proving the non-integrability in this case.

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1. Introduction

The spring-pendulum, which is also known under other names, such as swinging spring or elastic pendulum, is a very simple mechanical system having a very complex dynamical behaviour and this is why it is sometimes included in nonlinear paradigms. It consists of a point with mass m suspended from a fixed point by a light spring, moving under a constant vertical gravitation field. In Cartesian coordinates (x, y, z) with the origin at the point of suspension of the pendulum, the system is described by the following Hamiltonian:

$$H_0 = \frac{1}{2m}\left(p_x^2 + p_y^2 + p_z^2\right) + mgz + \frac{1}{2}k(r - l_0)^2 \tag{1}$$

where $r = \sqrt{x^2 + y^2 + z^2}$, l_0 is the unstretched length of the spring and $k \in \mathbb{R}^+$ is the Young modulus of the spring. The motion of this system is a complicated combination of two motions: swinging like a pendulum and bouncing up and down like a spring.

According to our knowledge, this system appeared first in [1] as a simple classical analogue for the quantum phenomenon of Fermi resonance in the infra-red spectrum of carbon dioxide. More about the history of this system can be found in [2]. Recently it has been analysed in connection with the modelling of phenomena in the atmosphere [3–5]. Because of the complicated dynamics, various approaches for its analysis were applied: asymptotic methods [6], various perturbation methods [7, 8], numerical methods [9], various formulations of KAM theorem, the Poincaré section, the Lapunov exponents [10], the Melnikov method [11, 12], etc. A brief review of a large number of earlier papers on the spring-pendulum can be found in [3] and [13].

The Hamiltonian system generated by (1) possesses two first integrals: Hamilton function H_0 and the third component of the angular momentum

$$p_z = x\dot{y} - y\dot{x}$$

and for its complete integrabilty in the Liouville sense, the third first integral is missing. Numerical computations suggest that such an additional first integral does not exist and the system is chaotic. The first rigorous non-integrability proof for this system was obtained by Churchill *et al* [14] by means of the Ziglin theory [15, 16]. This result can be formulated in the following theorem:

Theorem 1. *If the Hamiltonian system given by Hamiltonian function H_0 is integrable with meromorphic first integrals in the Liouville sense, then*

$$k = \frac{1-q^2}{q^2-9} \tag{2}$$

where q is a rational number.

Morales-Ruiz and Ramis using their theory formulated in [17] obtained a stronger result; they restricted the family (2) of values of parameter k for which the system can be integrable. Namely, they proved in [18] the following:

Theorem 2. *If the Hamiltonian system given by Hamiltonian function H_0 is integrable with meromorphic first integrals in the Liouville sense, then*

$$k = -\frac{p(p+1)}{p^2+p-2} \tag{3}$$

where p is an integer.

From the above theorem it easily follows that the physical spring-pendulum with $k \in \mathbb{R}^+$ is non-integrable except for the case $k = 0$. For $k = 0$, the system is integrable because of separation of variables in the potential.

In fact, the results presented above concern the two-dimensional spring-pendulum system obtained in the following way. If we choose initial conditions in such a way that the value of p_z equals zero, then the motion takes place in a vertical plane and we obtain the two-dimensional system. But the non-integrability of the two-dimensional spring-pendulum implies immediately the non-integrability of the three-dimensional spring-pendulum.

The aim of this paper is to investigate the integrability of the spring-pendulum system when the elastic potential also contains a cubic term. In other words, we consider a generalized spring-pendulum system described by the following Hamiltonian:

$$H = \frac{1}{2m}\left(p_x^2 + p_y^2 + p_z^2\right) + mgz + \frac{1}{2}k(r-l_0)^2 - \frac{1}{3}a(r-l_0)^3 \tag{4}$$

where $k \in \mathbb{R}^+$ and $a \in \mathbb{R}$. Our main result is the following:

Theorem 3. *If the Hamiltonian system given by Hamiltonian function (4) is integrable with meromorphic first integrals in the Liouville sense, then $k = -a$.*

In our proof of this theorem, we apply the Morales-Ruiz–Ramis theory and some tools of differential algebra. Basic facts from the Morales-Ruiz–Ramis theory and some results concerning special linear differential equations are presented in section 2. We derive variational equations and the normal variational equations for a family of particular solutions in section 3. Theorem 3 is proved in sections 4 (case $a = 0$) and 5 (case $a \neq 0$). In section 4, we revise the result of Morales-Ruiz formulated in theorem 2. Namely, we show that for values of k given by condition (3), the system is non-integrable except for the case $k = 0$. In this section we also show two different kinds of arguments which give rise to non-integrability of the classical spring-pendulum system when $k \geqslant 0$. In section 6, we study the exceptional case $a = -k$ and conclude that the Morales-Ruiz–Ramis method yields no obstruction to integrability, whereas dynamical analysis seems to indicate that the system is not completely integrable.

The Morales-Ruiz–Ramis theory was applied to study the integrability of many Hamiltonian systems, see examples in the book [17] and in papers [19–28]. The differential Galois approach was also used for proving non-integrability of non-Hamiltonian systems, see [29–31]. Difficulties in application of this theory can be of a different nature but mainly depend on the dimensionality of the problem and the number of parameters. Although it seems that the Morales-Ruiz–Ramis theory gives the strongest necessary conditions for the integrability, as far as we know, no new integrable system was found with the help of it. For the system investigated in this paper we have a very exceptional situation: we found a one parameter family of Hamiltonian systems for which the necessary conditions of integrability are satisfied, but, nevertheless, there is evidence that this family is not integrable, either. Another example of such a family can be found in [28].

2. Theory

Below we only mention basic notions and facts concerning the Ziglin and the Morales-Ruiz–Ramis theory following [15, 17, 32]:

Let us consider a system of differential equations

$$\frac{\mathrm{d}}{\mathrm{d}t}x = v(x) \qquad t \in \mathbb{C} \qquad x \in M \tag{5}$$

defined on a complex n-dimensional manifold M. If $\varphi(t)$ is a non-equilibrium solution of (5), then the maximal analytic continuation of $\varphi(t)$ defines a Riemann surface Γ with t as a local coordinate. Together with system (5), we can also consider variational equations (VEs) restricted to $T_\Gamma M$, i.e.

$$\dot{\xi} = T(v)\xi \qquad T(v) = \frac{\partial v}{\partial x} \qquad \xi \in T_\Gamma M. \tag{6}$$

We can always reduce the order of this system by one considering the induced system on the normal bundle $N := T_\Gamma M / T\Gamma$ of Γ [33]

$$\dot{\eta} = \pi_\star(T(v)\pi^{-1}\eta) \qquad \eta \in N. \tag{7}$$

Here $\pi : T_\Gamma M \to N$ is the projection. The system of $s = n - 1$ equations obtained in this way yields the so-called normal variational equations (NVEs). The monodromy group $\mathcal{M}$ of system (7) is the image of the fundamental group $\pi_1(\Gamma, t_0)$ of Γ obtained in the process of continuation of local solutions of (7) defined in a neighbourhood of t_0 along closed paths with the base point t_0. By definition it is obvious that $\mathcal{M} \subset GL(s, \mathbb{C})$. A non-constant rational function $f(z)$ of s variables $z = (z_1, \ldots, z_s)$ is called an integral (or invariant) of the monodromy group if $f(g \cdot z) = f(z)$ for all $g \in \mathcal{M}$.

In his two fundamental papers [15, 16], Ziglin showed that if system (5) possesses a meromorphic first integral, then the monodromy group $\mathcal{M}$ of the normal variational equations (7) has a rational integral (an invariant function). This result allowed him to formulate a necessary condition for the integrability of Hamiltonian systems.

If system (5) is Hamiltonian then necessarily $n = 2m$ and we have one first integral, namely the Hamiltonian H of the system. For a given particular solution $\varphi(t)$ we fix the energy level $E = H(\varphi(t))$. Restricting (5) to this level, we obtain a well-defined system on the $(n-1)$ dimensional manifold with a known particular solution $\varphi(t)$. For this restricted system we perform the reduction of order of variational equations. Thus, the normal variational equations for a Hamiltonian system with m degrees of freedom have dimension $s = 2(m-1)$ and their monodromy group is a subgroup of $\mathrm{Sp}(s, \mathbb{C})$. The spectrum of an element of the monodromy group $g \in \mathcal{M} \subset \mathrm{Sp}(2(m-1), \mathbb{C})$ has the form

$$\mathrm{spectr}(g) = \left(\lambda_1, \lambda_1^{-1}, \ldots, \lambda_{m-1}, \lambda_{m-1}^{-1}\right) \qquad \lambda_i \in \mathbb{C}$$

and g is called resonant if

$$\prod_{l=1}^{m-1} \lambda_l^{k_l} = 1 \qquad \text{for some} \quad (k_1, \ldots, k_{m-1}) \in \mathbb{Z}^{m-1} \quad \sum_{i=1}^{m-1} k_i \neq 0.$$

In [15] Ziglin proved the main theorem of his theory. Here we formulate it as in [33].

Theorem 4. *Let us assume that there exists a non-resonant element $g \in \mathcal{M}$. If the Hamiltonian system with m degrees of freedom has m meromorphic first integrals $F_1 = H, \ldots, F_m$, which are functionally independent in a connected neigbourhood of Γ, then any other monodromy matrix $g' \in \mathcal{M}$ transforms eigenvectors of g to its eigenvectors.*

Recently Morales-Ruiz and Ramis generalized the Ziglin approach replacing the monodromy group $\mathcal{M}$ by the differential Galois group $\mathcal{G}$ of NVEs, see [17, 34]. For a detailed exposition of the differential Galois theory see [36, 35, 17]. Here we describe the notion of the differential Galois group for a second-order linear differential equation with rational coefficients:

$$w'' + pw' + qw = 0 \qquad p, q \in \mathbb{C}(t) \qquad {}' \equiv \frac{\mathrm{d}}{\mathrm{d}t}. \tag{8}$$

In what follows we keep algebraic notation, e.g., by $\mathbb{C}[t]$ we denote the ring of polynomials of one variable t, $\mathbb{C}(t)$ is the field of rational functions, etc. Here we consider the field $\mathbb{C}(t)$ as a differential field, i.e. a field with distinguished differentiation. Note that in our case all elements $a \in \mathbb{C}(t)$ such that $a' = 0$ are just constant, i.e. we have $a' = 0 \Leftrightarrow a \in \mathbb{C}$. Thus such elements form a field—the field of constants.

Remark 1. In the most general case we meet in applications, the coefficients of (6) are meromorphic functions defined on a Riemann surface Γ, which is usually denoted by $\mathcal{M}(\Gamma)$. Meromorphic functions on Γ form a field. It is a differential field if it is equipped with ordinary differentiation.

The field $\mathbb{C}(t)$ can be extended to a larger differential field K such that it will contain all solutions of equation (8). The smallest differential field K containing n linearly independent solutions of (8) is called the Picard–Vessiot extension of $\mathbb{C}(t)$ (additionally we need the field of constants of K to be $\mathbb{C}$).

The Picard–Vessiot extension for equation (8) can be constructed in the following way. We take two linearly independent solutions ξ and η of (8) (we know that such solutions exist). Then, as K we take all rational functions of five variables $(t, \xi, \xi', \eta, \eta')$, i.e. $K = \mathbb{C}(t, \xi, \xi', \eta, \eta')$.

The differential Galois group of equation (8) is defined as follows. For the Picard–Vessiot extension $K \supset \mathbb{C}(t)$, we consider all automorphisms of K (i.e. invertible transformations of K preserving field operations) which commute with differentiation. An automorphism $g : K \to K$ commutes with differentiation if $g(v') = (g(v))'$, for all $v \in K$. We denote by $\mathcal{A}$ the set of all such automorphisms. Let us note that automorphisms $\mathcal{A}$ form a group. The differential Galois group $\mathcal{G}$ of extension $K \supset \mathbb{C}(t)$, is, by definition, a subgroup of $\mathcal{A}$ such that it contains all automorphisms g which do not change elements of $\mathbb{C}(t)$, i.e. for $g \in \mathcal{G}$ we have $g(v) = v$ for all $v \in \mathbb{C}(t)$.

It seems that the definition of the differential Galois group is abstract and that it is difficult to work with it. However, from this definition we can deduce that it can be considered as a subgroup of invertible matrices. Let $\mathcal{G}$ be the differential Galois group of equation (8) and let $g \in \mathcal{G}$. Then we have

$$0 = g(0) = g(w'' + pw' + qw) = g(w'') + g(p)g(w') + g(q)g(w)$$

but g commutes with differentiation so $g(w'') = (g(w))''$, $g(w') = (g(w))'$, and, moreover, $g(p) = p$, $g(q) = q$ because $p, q \in \mathbb{C}(t)$. Thus we have

$$(g(w))'' + p(g(w))' + qg(w) = 0.$$

In other words, if w is a solution of equation (8) then $g(w)$ is also its solution. Thus, if ξ and η are linearly independent solutions of (8), then

$$g(\xi) = g_{11}\xi + g_{21}\eta \qquad g(\eta) = g_{12}\xi + g_{22}\eta$$

and

$$g\left(\begin{bmatrix} \xi & \eta \\ \xi' & \eta' \end{bmatrix}\right) = \begin{bmatrix} \xi & \eta \\ \xi' & \eta' \end{bmatrix}\begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix}.$$

Hence, we can associate with an element g of the differential Galois group $\mathcal{G}$ an invertible matrix $[g_{ij}]$, and thus we can consider $\mathcal{G}$ a subgroup of $GL(2, \mathbb{C})$. If instead of the solutions ξ and η we take two other linearly independent solutions, then all matrices $[g_{ij}]$ are changed by the same similarity transformation.

The construction presented above can be easily generalized to a linear differential equation of an arbitrary order n and to a system of linear equations. Thus, we can treat the differential Galois group as a subgroup of $GL(n, \mathbb{C})$. Let us list basic facts about the differential Galois group

(i) If $g(v) = v$ for all $g \in \mathcal{G}$, then $v \in \mathbb{C}(t)$.
(ii) Group $\mathcal{G}$ is an algebraic subgroup of $GL(n, \mathbb{C})$. Thus it has a unique connected component $\mathcal{G}^0$ which contains the identity, and which is a normal subgroup of finite index.
(iii) Every solution of the differential equation is Liouvillian if and only if $\mathcal{G}^0$ conjugates to a subgroup of the triangular group. This is the Lie–Kolchin theorem.

Remark 2. In the case considered (a system of complex linear equations with rational coefficients), the existence of the Picard–Vessiot extension follows from the Cauchy existence theorem. In abstract settings, i.e. when we consider a differential equation with coefficients in an abstract differential field, the existence of the Picard–Vessiot extension is a non-trivial fact, see, e.g., [37].

Certain differential extensions of $\mathbb{C}(t)$ are important in applications because they define what we understand by a solvable equation. We say that element η of a differential extension $K \supset \mathbb{C}(t)$ is

(i) *algebraic* over $\mathbb{C}(t)$ if η satisfies a polynomial equation with coefficients in $\mathbb{C}(t)$,

(ii) *primitive* over $\mathbb{C}(t)$ if $\eta' \in \mathbb{C}(t)$, i.e. if $\eta = \int a$, for certain $a \in \mathbb{C}(t)$,
(iii) *exponential* over $\mathbb{C}(t)$ if $\eta'/\eta \in \mathbb{C}(t)$, i.e. if $\eta = \exp \int a$, for certain $a \in \mathbb{C}(t)$.

We say that a differential field L is a Liouvillian extension of $\mathbb{C}(t)$ if it can be obtained by successive extensions

$$\mathbb{C}(t) = K_0 \subset K_1 \subset \cdots \subset K_m = L$$

such that $K_i = K_{i-1}(\eta_i)$ with η_i either algebraic, primitive or exponential over K_{i-1}. We say that (6) is solvable if for it the Picard–Vessiot extension is a Liouvillian extension.

Remark 3. All elementary functions, such as e^t, $\log t$, trigonometric functions, are Liouvillian, but special functions such as Bessel or Airy functions are not Liouvillian.

Morales-Ruiz and Ramis formulated a new criterion of the non-integrability for Hamiltonian systems in terms of the properties of the identity component $\mathcal{G}^0$ of the differential Galois group of the normal variational equations, see [17, 34].

Theorem 5. *Assume that a Hamiltonian system is meromorphically integrable in the Liouville sense in a neighbourhood of the analytic curve* Γ. *Then the identity component of the differential Galois group of NVEs associated with* Γ *is Abelian.*

We see that the assumptions in the above theorem are stronger than in the Ziglin theorem. Moreover, as $\mathcal{G} \supset \mathcal{M}$, theorem 5 gives stronger necessary integrability conditions than the Ziglin criterion.

In applications of the Morales-Ruiz–Ramis criterion the first step is to find a non-equilibrium particular solution, very often it lies on an invariant submanifold. Next, we calculate VEs and NVEs. In the last step we have to check if $\mathcal{G}^0$ of obtained NVEs is Abelian. Very often in applications we check only if $\mathcal{G}^0$ is solvable, because if it is not, then the system is not integrable.

For some systems the necessary conditions for the integrability formulated in theorem 5 are satisfied, but, nevertheless, they are non-integrable. In such cases, to prove the non-integrability we can use the stronger version of the Morales-Ruiz–Ramis theorem based on higher order variational equations [17, 38]. The idea of higher variational equations is the following. For system (5) with a particular solution $\varphi(t)$ we put

$$x = \varphi(t) + \varepsilon\xi^{(1)} + \varepsilon^2\xi^{(2)} + \cdots + \varepsilon^k\xi^{(k)} + \cdots$$

where ε is a formal small parameter. Inserting the above expansion into equation (5) and comparing terms of the same order with respect to ε, we obtain the following chain of linear inhomogeneous equations:

$$\frac{\mathrm{d}}{\mathrm{d}t}\xi^{(k)} = \mathbf{A}(t)\xi^{(k)} + f_k(\xi^{(1)}, \ldots, \xi^{(k-1)}) \qquad k = 1, 2, \ldots \tag{9}$$

where

$$\mathbf{A}(t) = \frac{\partial v}{\partial x}(\varphi(t))$$

and $f_1 \equiv 0$. For a given k, equation (9) is called the kth order variational equation (VE$_k$). Note that VE$_1$ coincides with (6). There is an appropriate framework allowing us to define the differential Galois group of kth order variational equations, for details see [17, 38]. The following theorem was announced in [38]:

Theorem 6. *Assume that a Hamiltonian system is meromorphically integrable in the Liouville sense in a neighbourhood of the analytic curve* Γ. *Then the identity components of the* *differential Galois group of the kth order variational equations* VE_k *are Abelian for any* $k \in \mathbb{N}$.

There is also a possibility that the differential Galois groups of an arbitrary order variational equation are Abelian. Then we have to use another particular solution for the non-integrability proof.

For further considerations we need some known facts about linear differential equations of special forms. First we consider the Riemann P equation [39]

$$\frac{\mathrm{d}^2\eta}{\mathrm{d}z^2} + \left(\frac{1-\alpha-\alpha'}{z} + \frac{1-\gamma-\gamma'}{z-1}\right)\frac{\mathrm{d}\eta}{\mathrm{d}z} + \left(\frac{\alpha\alpha'}{z^2} + \frac{\gamma\gamma'}{(z-1)^2} + \frac{\beta\beta'-\alpha\alpha'-\gamma\gamma'}{z(z-1)}\right)\eta = 0 \tag{10}$$

where (α, α'), (γ, γ') and (β, β') are the exponents at singular points. They satisfy the Fuchs relation

$$\alpha + \alpha' + \gamma + \gamma' + \beta + \beta' = 1.$$

We denote the differences of exponents by

$$\lambda = \alpha - \alpha' \qquad \nu = \gamma - \gamma' \qquad \mu = \beta - \beta'.$$

Necessary and sufficient conditions for solvability of the identity component of the differential Galois group of (10) are given by the following theorem due to Kimura [40], see also [17].

Theorem 7. *The identity component of the differential Galois group of equation (10) is solvable if and only if*

(A) at least one of the four numbers $\lambda + \mu + \nu$, $-\lambda + \mu + \nu$, $\lambda - \mu + \nu$, $\lambda + \mu - \nu$ *is an odd integer, or*

(B) the numbers λ *or* $-\lambda$ *and* μ *or* $-\mu$ *and* ν *or* $-\nu$ *belong (in an arbitrary order) to some of the following 15 families:*

1	$\frac{1}{2}+l$	$\frac{1}{2}+m$	*Arbitrary complex number*	
2	$\frac{1}{2}+l$	$\frac{1}{3}+m$	$\frac{1}{3}+q$	
3	$\frac{2}{3}+l$	$\frac{1}{3}+m$	$\frac{1}{3}+q$	$l+m+q$ *even*
4	$\frac{1}{2}+l$	$\frac{1}{3}+m$	$\frac{1}{4}+q$	
5	$\frac{2}{3}+l$	$\frac{1}{4}+m$	$\frac{1}{4}+q$	$l+m+q$ *even*
6	$\frac{1}{2}+l$	$\frac{1}{3}+m$	$\frac{1}{5}+q$	
7	$\frac{2}{5}+l$	$\frac{1}{3}+m$	$\frac{1}{3}+q$	$l+m+q$ *even*
8	$\frac{2}{3}+l$	$\frac{1}{5}+m$	$\frac{1}{5}+q$	$l+m+q$ *even*
9	$\frac{1}{2}+l$	$\frac{2}{5}+m$	$\frac{1}{5}+q$	$l+m+q$ *even*
10	$\frac{3}{5}+l$	$\frac{1}{3}+m$	$\frac{1}{5}+q$	$l+m+q$ *even*
11	$\frac{2}{5}+l$	$\frac{2}{5}+m$	$\frac{2}{5}+q$	$l+m+q$ *even*
12	$\frac{2}{3}+l$	$\frac{1}{3}+m$	$\frac{1}{5}+q$	$l+m+q$ *even*
13	$\frac{4}{5}+l$	$\frac{1}{5}+m$	$\frac{1}{5}+q$	$l+m+q$ *even*
14	$\frac{1}{2}+l$	$\frac{2}{5}+m$	$\frac{1}{3}+q$	$l+m+q$ *even*
15	$\frac{3}{5}+l$	$\frac{2}{5}+m$	$\frac{1}{3}+q$	$l+m+q$ *even*

Here l, m *and* q *are integers.*

Next we consider the Lamé equation in the standard Weierstrass form

$$\frac{\mathrm{d}^2\xi}{\mathrm{d}t^2} = [n(n+1)\wp(t) + B]\xi \tag{11}$$

where n and B are, in general, complex parameters and $\wp(t)$ is the elliptic Weierstrass function with invariants g_2, g_3. In other words, $\wp(t)$ is a solution of differential equation

$$\dot{x}^2 = f(x) \qquad f(x) := 4x^3 - g_2 x - g_3 = 4(x - x_1)(x - x_2)(x - x_3). \tag{12}$$

We assume that parameters n, B, g_2 and g_3 are such that

$$\Delta = g_2^3 - 27g_3^2 \neq 0$$

and thus equation $f(x) = 0$ has three different roots x_1, x_2 and x_3. All the cases when the Lamé equation is solvable are listed in the following theorem, see [17]:

Theorem 8. *The Lamé equation is solvable only in the following cases:*

(i) the Lamé and Hermite case (see, e.g., [41]) for which $n \in \mathbb{Z}$ and three other parameters are arbitrary,
(ii) the Brioschi–Halphen–Crowford case (see, e.g., [42, 41]). In this case $n + \frac{1}{2} \in \mathbb{N}$ and B, g_2, g_3 satisfy an appropriate algebraic equation,
(iii) the Baldassarri case [42]. Then $n + \frac{1}{2} \in \frac{1}{3}\mathbb{Z} \cup \frac{1}{4}\mathbb{Z} \cup \frac{1}{5}\mathbb{Z} \backslash \mathbb{Z}$, and there are additional algebraic conditions on B, g_2, g_3.

We consider the second-order differential equation

$$\eta'' + p(z)\eta' + q(z)\eta = 0 \qquad ' = \frac{\mathrm{d}}{\mathrm{d}z} \qquad p(z), q(z) \in \mathbb{C}(z).$$

Putting

$$\eta = y \exp\left[-\frac{1}{2}\int_{z_0}^{z} p(s)\,\mathrm{d}s\right]$$

we obtain its reduced form

$$y'' = r(z)y \qquad r(z) = -q(z) + \tfrac{1}{2}p'(z) + \tfrac{1}{4}p(z)^2. \tag{13}$$

For this equation its differential Galois group $\mathcal{G}$ is an algebraic subgroup of $SL(2, \mathbb{C})$. The following theorem describes all possible forms of $\mathcal{G}$ and relates them to forms of solutions of (13), see [43, 17].

Lemma 1. *Let $\mathcal{G}$ be the differential Galois group of equation (13). Then one of four cases can occur.*

(i) $\mathcal{G}$ is conjugated to a subgroup of the triangular group; in this case equation (13) has a solution of the form $y = \exp\int\omega$, where $\omega \in \mathbb{C}(z)$,
(ii) $\mathcal{G}$ is conjugated with a subgroup of

$$D^{\dagger} = \left\{\begin{bmatrix} c & 0 \\ 0 & c^{-1}\end{bmatrix} \middle| c \in \mathbb{C}^*\right\} \cup \left\{\begin{bmatrix} 0 & c \\ c^{-1} & 0\end{bmatrix} \middle| c \in \mathbb{C}^*\right\}$$

in this case equation (13) has a solution of the form $y = \exp\int\omega$, where ω is algebraic over $\mathbb{C}(z)$ of degree 2,
(iii) $\mathcal{G}$ is primitive and finite; in this case all solutions of equation (13) are algebraic,
(iv) $\mathcal{G} = SL(2, \mathbb{C})$ and equation (13) has no Liouvillian solution.

An equation with a Liouvillian solution is called integrable. When case (i) in the above lemma occurs, we say that the equation is reducible and its solution of the form prescribed for this case is called exponential.

Remark 4. Let us assume that equation (13) is Fuchsian, i.e. $r(z)$ has poles at $z_i \in \mathbb{C}$, $i = 1, \ldots, K$ and at $z_{K+1} = \infty$; all of them are of order not higher than 2. Then at each singular point z_i and $z = \infty$ we have two (not necessarily different) exponents, see, e.g., [39]. One can show, see [43], that an exponential solution which exists when case (i) in lemma 1 occurs, has the following form:

$$y = P \prod_{i=1}^{K} (z - z_i)^{e_i}$$

where e_i is an exponent at the singular point; P is a polynomial and, moreover

$$\deg P = -e_\infty - \sum_{i=1}^{K} e_i$$

where e_∞ is an exponent at infinity.

Remark 5. If equation (13) has a regular singular point z_0 with exponents (e_1, e_2) and $e_1 - e_2 \notin \mathbb{Z}$, then in a neighbourhood of z_0 there exist two linearly independent solutions of the form

$$y_i = (z - z_0)^{e_i} f_i(z) \qquad i = 1, 2$$

where $f_i(z)$ are holomorphic at z_0. If $e_1 - e_2 \in \mathbb{Z}$, then one local solution has the above form (for the exponent with a larger real part). The second solution can contain a logarithmic term, for details see [39]. If the logarithmic term appears, then it can be shown that only case (i) and case (iv) in lemma 1 can occur, see [23].

3. Particular solution and variational equations

Without loss of generality, choosing appropriately units of time, mass and length, we can put $m = g = l_0 = 1$. Then the Hamiltonian of the generalized spring-pendulum in spherical coordinates has the following form:

$$H = \frac{1}{2}\left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\varphi^2}{r^2 \sin^2\theta}\right) - r\cos\theta + \frac{k}{2}(r-1)^2 - \frac{a}{3}(r-1)^3 .$$

As we can see φ is a cyclic coordinate and p_φ is a first integral. Manifold

$$\mathcal{N} = \{(r, \theta, \varphi, p_r, p_\theta, p_\varphi) \in \mathbb{C}^6 \mid \theta = \varphi = p_\theta = p_\varphi = 0\}$$

is invariant with respect to the flow of Hamilton equations generated by H. Hamiltonian equations restricted to $\mathcal{N}$ have the form

$$\dot{r} = p_r \qquad \dot{p}_r = 1 - k(r-1) + a(r-1)^2$$

and can be rewritten as

$$\ddot{r} = 1 - k(r-1) + a(r-1)^2.$$

Thus the phase curve located on the energy level $H|_{\mathcal{N}} = E$ is given by the equation

$$E = \frac{\dot{r}^2}{2} + \frac{k}{2}(r-1)^2 - \frac{a}{3}(r-1)^3 - r \tag{14}$$

and hence, for the generic values of E, it is an elliptic curve when $a \neq 0$ (for $a = 0$ it is a sphere). To find its explicit time parametrization we put

$$r = \frac{6}{a}x + \frac{2a+k}{2a} \tag{15}$$

then (14) transforms into an equation of the form (12) with

$$g_2 = \frac{k^2 - 4a}{12} \qquad g_3 = \frac{k^3 - 6ak - 12a^2(E+1)}{216}. \tag{16}$$

For these invariants, $x(t)$ is a non-degenerate Weierstrass function provided that

$$\Delta = \frac{(4a-k^2)^3 + [k^3 - 6ak - 12a^2(E+1)]^2}{1728} \neq 0.$$

But $\Delta = 0$ only for two exceptional values of energy corresponding to unstable and stable equilibria (we assume here that $a \neq 0$):

$$\begin{aligned} E_{\mathrm{u}} &= \frac{k^3 - 6a(k+2a) + (k^2-4a)^{3/2}}{12a^2} \\ E_{\mathrm{s}} &= \frac{k^3 - 6a(k+2a) - (k^2-4a)^{3/2}}{12a^2}. \end{aligned} \tag{17}$$

For $E_{\mathrm{s}} < E < E_{\mathrm{u}}$, we obtain one parameter family $\Gamma(t, E)$ of particular solutions $(r(t), 0, 0, p_r(t), 0, 0)$ expressed in terms of the Weierstrass function and its derivative as

$$r(t) = \frac{6}{a}\wp(t; g_2, g_3) + 1 + \frac{k}{2a} \qquad p_r(t) = \frac{6}{a}\dot{\wp}(t; g_2, g_3). \tag{18}$$

Particular solutions are single-valued, meromorphic and double periodic with periods $2\omega_1$ and $2\omega_2$, and they have one double pole at $t = 0$. Thus, Riemann surfaces $\Gamma(t, E)$ are tori with one point removed.

Using first integral p_φ, we can reduce the order of VEs by two. We choose the zero level of this first integral. Let $\eta = (R, P_R, \Theta, P_\Theta)$ denote variations in $(r, p_r, \theta, p_\theta)$. Then the reduced variational equations restricted to the level $p_\varphi = 0$ have the form

$$\frac{\mathrm{d}\eta}{\mathrm{d}t} = \mathbf{L}\eta$$

where matrix $\mathbf{L}$ is given by

$$\mathbf{L} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 2a(r-1)-k & 0 & 0 & 0 \\ 0 & 0 & 0 & r^{-2} \\ 0 & 0 & -r & 0 \end{bmatrix}. \tag{19}$$

The normal variational equations read

$$\dot{\Theta} = \frac{1}{r^2}P_\Theta \qquad \dot{P}_\Theta = -r\Theta$$

and can be written as

$$\ddot{\Theta} + 2\frac{\dot{r}(t)}{r(t)}\dot{\Theta} + \frac{1}{r(t)}\Theta = 0. \tag{20}$$

Putting $\Phi = \Theta r$, and expressing r in terms of the Weierstrass function using (18) we transform (20) to the form

$$\ddot{\Phi} + \frac{k^2 - 144\wp(t)^2}{24\wp(t) + 2k + 4a}\Phi = 0. \tag{21}$$

Apart from $t = 0$, equation (21) has other singular points which are solutions of the equation

$$\wp(t) = d := -\tfrac{1}{12}(k + 2a).$$

If $d \notin \{x_1, x_2, x_3\}$, then the above equation has two roots. If $d = x_k$, $k = 1, 2, 3$, then this equation has one double root. If $a = -k$, then $t = 0$ is the only singular point and in this case (21) is the Lamé equation.

4. Non-integrability of the classical spring-pendulum

In this section we investigate the classical spring-pendulum, i.e. we assume that $a = 0$. In this case Hamiltonian equations restricted to manifold $\mathcal{N}$ have the form

$$\dot{r} = p_r \qquad \dot{p}_r = 1 - k(r - 1)$$

and the phase curve corresponding to energy value E is a sphere

$$E = \frac{\dot{r}^2}{2} + \frac{k}{2}(r - 1)^2 - r.$$

Making transformation $t \mapsto z = r(t)$ we transform NVE (20) to a Fuchsian equation with rational coefficients and four singular points $z_0 = 0$, $z_1 = z_1(E)$, $z_2 = z_2(E)$ and $z_3 = \infty$, i.e. for generic values of E the transformed NVE is a Heun equation. However, changing E, we are able to make a confluence of two singular points and for these special choices of E the transformed NVE has the form of the Riemann P equation (10). We have two possibilities: we can chose $E = E_1$ such that $z_1(E_1) = z_2(E_1)$, or we can take $E = E_2$ such that $z_1(E_2) = 0$. In both cases we obtain a Riemann P equation, however, these two Riemann equations are not equivalent and thus they give two *different* necessary conditions for the integrability. It seems that this fact was not noted in previous investigations.

Let us assume that $k \neq 0$ and put $E = -(2k + 1)/(2k)$. Then the following change of variable:

$$t \mapsto z := \frac{k}{1 + k} r(t)$$

transforms (20) to the form

$$y'' + \left(\frac{2}{z} + \frac{1}{z - 1}\right) y' + \left(-\frac{1}{(1 + k)(z - 1)^2} + \frac{1}{(1 + k)z(z - 1)}\right) y = 0 \qquad (22)$$

where $y = y(z) := \Theta(t(z))$. This Riemann P equation has exponents

$$\alpha = 0 \qquad \alpha' = -1 \qquad \beta = 2 \qquad \beta' = 0 \qquad \gamma = -\gamma' = \frac{1}{\sqrt{1 + k}}.$$

The prescribed choice of the energy corresponds to E_1, i.e. in the generic Heun equation two non-zero singular points collapse to one. We prove the following:

Lemma 2. *If $k \neq 0$ and*

$$k \neq \frac{1}{(m + 2)^2} - 1$$

where m is a non-negative integer, then equation (22) does not possess a Liouvillian solution.

Proof. Local computation shows that equation (22) has logarithms in its formal solutions at zero and infinity whenever $k \neq 0$. Thus, as we know from remark 5, if the equation has a Liouvillian solution, then it must be an exponential one, i.e. we are in case (i) of lemma 1. As the equation is Fuchsian, from remark 4 it follows that such an exponential solution has the form

$$y = z^{e_0}(z - 1)^{e_1} P(z)$$

where e_i is an exponent at $z = i, i = 0, 1$, and P is a polynomial whose degree m satisfies $m = -e_\infty - e_0 - e_1$. Moreover, an expansion of an exponential solution of the form given above around a singular point does not contain logarithms. However, we know that there are formal solutions at $z = 0$ and $z = \infty$ with logarithms. Thus those without logarithms correspond to the maximal exponents, see [39]. Hence, we must put $e_0 = \alpha = 0$, $e_\infty = \beta = 2$, and we may take $e_1 = \gamma = 1/\sqrt{1+k}$. The condition on the degree of P imposes that

$$k = \frac{1}{(m+2)^2} - 1$$

with m a non-negative integer. As we excluded such values of k this finishes the proof. □

Now, for all non-negative integers m, we have $(m+2)^{-2} - 1 < 0$ so, as for a physical spring we have $k > 0$, the above lemma shows that equation (22) has no exponential solution (which was the only possible integrable case) and, finally, the NVE is not integrable. This ends the proof of theorem 3 for case $a = 0$.

Remark 6. Of course we can prove lemma 2 using theorem 7. For equation (22) differences of exponents are

$$\lambda = 1 \qquad \nu = \frac{2}{\sqrt{1+k}} \qquad \mu = 2.$$

It is easy to note that case B in the Kimura theorem is impossible. Thus equation (22) is solvable (i.e. the identity component of the differential Galois group is solvable) if and only if the condition from case A of the Kimura theorem is satisfied. The four numbers from case A of the Kimura theorem are equal to

$$\lambda + \mu + \nu = 3 + \frac{2}{\sqrt{1+k}} \qquad -\lambda + \mu + \nu = 1 + \frac{2}{\sqrt{1+k}}$$
$$\lambda - \mu + \nu = -1 + \frac{2}{\sqrt{1+k}} \qquad \lambda + \mu - \nu = 3 - \frac{2}{\sqrt{1+k}}.$$

The condition that at least one of them is an odd integer is equivalent to $k = (m+2)^{-2} - 1$. We gave another proof of lemma 2 in order to demonstrate a technique which we use in the next section.

To appreciate the relevance of the physical hypothesis $k > 0$, we prove the following:

Lemma 3. *If $k = (m+2)^{-2} - 1$ with m a non-negative integer, then the identity component of the differential Galois group of equation (22) is Abelian.*

Proof. Proceeding as in the proof of lemma 2, we conclude that under our assumption equation (22) is solvable if and only if it has a solution of the form $y = P/(z-1)^{m+2}$ with P a polynomial of degree m. Following the method of [23], we make the change of variables $y = Y/(z-1)^{m+2}$ in (22) and compute the recurrence relation satisfied by the coefficients of a power series solution $Y = \sum u_n z^n$ at zero. The recurrence is

$$(n-m)(n-2-m)u_n = (n+1)(n+2)u_{n+1}.$$

The latter always admits a solution such that $u_{-1} = u_{m+1} = 0$, $u_0 = 1$ and $u_m \neq 0$, which proves that for all non-negative integers m, the NVE with $k = (m+2)^{-2} - 1$ admits a solution of the form $y = P/(z-1)^{m+2}$ with P a polynomial of degree m. Thus, the differential Galois group of equation (22) conjugates to a subgroup of the triangular group (case (i) in lemma 1). Moreover, as all exponents are rational, its identity component is Abelian. □

The above lemma shows that when $k < 0$ (so, for the negative Young modulus) the necessary condition of the Morales-Ruiz–Ramis theory is satisfied for infinitely many cases. As integrable systems are extremely rare, it is worth checking if, even for non-physical values of k excluded in lemma 2, the system is integrable or not.

To answer this question we take $E = k/2$, and make the following change of variable:

$$t \mapsto z := \frac{kr(t)}{2(k+1)}$$

in equation (20). Choosing the prescribed value of energy, we perform a confluence of one non-zero singular point with $z = 0$ in the generic NVE, i.e. this energy corresponds to E_2. The NVE takes the following form:

$$y'' + \left(\frac{5}{2z} + \frac{1}{2(z-1)}\right) y' + \left(\frac{1}{2(1+k)(z-1)^2} - \frac{1}{2(1+k)z(z-1)}\right) y = 0. \tag{23}$$

This is exactly the form of NVE which appears in papers [14, 17, 18] and the condition for its non-integrability is given by (3). Combining the non-integrability conditions for equations (22) and (23) we show the following:

Theorem 9. *The classical spring-pendulum system given by Hamiltonian (1) with $k \in \mathbb{R}$ is integrable only when $k = 0$.*

Proof. Assume that the system is integrable. Then both NVEs (22) and (23) are integrable, i.e. they possess Liouvillian solutions. Thus, we have

$$k = \frac{1}{(m+2)^2} - 1$$

for some non-negative integer m, and

$$k = -\frac{p(p+1)}{p^2+p-2} \qquad p \in \mathbb{Z}.$$

But we can rewrite these conditions in the following form:

$$k = \frac{1-s}{s} \qquad s = (m-2)^2$$

and

$$k = \frac{r}{1-r} \qquad r = \frac{1}{2}p(p+1).$$

As we assumed that $k \neq 0$, both s and r are positive integers. Now, from equality

$$\frac{1-s}{s} = \frac{r}{1-r}$$

it follows that $r + s = 1$, but it is impossible for positive integers r and s. □

5. Non-integrability of the generalized spring-pendulum in the case $a \neq 0$ and $a \neq -k$

NVE given by (21) depends on the energy E through the invariants of the Weierstrass function, see formula (16). The choice of the value of energy is relevant for computation and we put

$$E = E_0 := \frac{2(3k+2a)a^2 - 1}{12a^2}.$$

For this value of the energy we have the following:

Lemma 4. *If $a \neq 0$ and $a \neq -k$, then the differential Galois group of the normalized NVE (21) for $E = E_0$ is equal to $SL(2, \mathbb{C})$.*

Proof. Computation shows that the image of equation (21) under the change of variable $t \mapsto x = \wp(t)$ is the following:

$$y''(x) + \frac{1}{2}\frac{f'(x)}{f(x)}y'(x) - \frac{144x^2 - k^2 - 2a(a+k)}{(12x + k + 2a)\, f(x)}y(x) = 0 \tag{24}$$

where

$$f(x) = 4x^3 - g_2 x - g_3 = 4(x - x_1)(x - x_2)(x - x_3)$$

g_2 and g_3 are given by (16) with $E = E_0$, and $y(x) = \Phi(t(x))$. This equation is Fuchsian and it has five singular points: $x_0 = -(2a + k)/12$, the three roots x_1, x_2, x_3 of $f(x)$ and $x_4 = \infty$. The exponents at the first singularity x_0 are $(0, 1)$, the exponents at the roots of $f(x)$ are $(0, 1/2)$ and the exponents at infinity are $(-1, 3/2)$.

If $a \notin \{0, -k\}$, then calculation of the formal solutions at x_0 shows that they contain a logarithm. So from remark 5 we know that the differential Galois group of equation (24) is either reducible or it is $SL(2, \mathbb{C})$.

Let us first assume that the equation is reducible (case (i) of lemma 1), i.e. it has an exponential solution. From remark 4 we know that such a solution has the form

$$y = P(x) \prod_{i=0}^{3} (x - x_i)^{e_i}$$

where the e_i is an exponent at $x = x_i$, and the degree m of polynomial $P(x)$ satisfies $m = -e_\infty - \sum_{i=0}^{3} e_i$.

Because the formal solution of valuation 0 at x_0 has a logarithm, the valuation (i.e. the exponent) of y at x_0 must be equal to 1, so $e_0 = 1$. The exponents at x_i for $i = 1, 2, 3$ of f are of the form $n_i/2$, (n_i a non-negative integer), so the relation for the degree m is either $m = -3/2 - 1 - n/2$, or $m = 1 - 1 + n/2$, for some non-negative integer $n = n_1 + n_2 + n_3$.

If $m = -3/2 - 1 - n/2$, then $m < 0$, which is not possible, so we must have $m = 1 - 1 - n/2 = -n/2$, which is possible only if $n = 0$, and hence $m = 0$.

So the only possibility is $y = (x - x_0)$. Substituting this candidate into equation (24) shows that this is not a solution. Hence, the equation is irreducible and, because of the logarithms in the local solutions, the only possibility is that the differential Galois group is the full $SL(2, \mathbb{C})$, which proves the lemma. □

Our main theorem 3 in the introduction now follows, as an immediate consequence of the Morales-Ruiz–Ramis theorem, from the considerations in section 4 (for the case $a = 0$) and from the above lemma.

6. Non-integrability of the generalized spring-pendulum in the case $a = -k$

First we show that for the excluded case $a = -k$, the necessary condition for integrability given by the Morales-Ruiz–Ramis theory is fulfilled.

Lemma 5. *For the generalized spring-pendulum in the case $a = -k$, the identity component of the differential Galois group of NVE (21) is Abelian.*

Proof. For $a = -k$, equation (21) reads

$$\ddot{\Phi} = \left(6\wp(t) + \tfrac{1}{2}k\right)\Phi \tag{25}$$

so it has the form of the Lamé equation (11) with $n(n+1) = 6$, and $B = k/2$. For the prescribed choice of parameters, the invariants g_2 and g_3 of the Weierstrass function $\wp(t)$ are the following:

$$g_2 = \frac{(k+4)k}{12} \qquad g_3 = \frac{k^2(k-12E-6)}{216}.$$

The discriminant

$$\Delta = \frac{k^3[k(k-12E-6)^2 - (k+4)^3]}{1728}$$

is only zero for two exceptional values of E corresponding to two local extrema of the potential. Assuming that E is different from these exceptional values, we can apply theorem 8.

As for equation (25) we have $n(n+1) = 6$, so $n = 2$ or $n = -3$. Since $n \in \mathbb{Z}$ Lamé equation (25) is solvable and possesses the Lamé–Hermite solutions [17, 41, 39]. But for a Lamé equation with such solutions the differential Galois group is Abelian [17].

For the excluded energy values, i.e. when $E = E_{\mathrm{s}}$ or $E = E_{\mathrm{u}}$ (see formula (17) the NVE (after transformation $t \mapsto z := r(t)$) has the form of Riemann P equation which is solvable. Namely, for both choices of E the case (ii) from lemma 1 occurs. Thus, the identity component of the differential Galois group of NVE is Abelian. $\square$

Let us note here that we have at our disposal another family of particular solutions corresponding to the following invariant manifold:

$$\mathcal{N}_1 = \{(r, \theta, \varphi, p_r, p_\theta, p_\varphi) \in \mathbb{C}^6 \mid \varphi = p_\theta = p_\varphi = 0, \theta = \pi\}.$$

However, as calculations show, using these particular solutions we do not obtain new necessary conditions for the integrability. Because of this, we decide to apply theorem 6.

Following the decoupling of the first VE into tangential and normal equations (see (19)), we find that the second variational equations are the following:

$$\ddot{r}_2 - 12\wp r_2 = \frac{8k^3(p_{\theta,1})^2}{(-12\wp + k)^3} - k\,(r_1)^2 - \frac{1}{2}\,(\theta_1)^2 \tag{26}$$

and

$$\begin{aligned}\ddot{\theta}_2 - \frac{24\dot{\wp}}{-12\wp + k}\dot{\theta}_2 + \frac{2k}{-12\wp + k}\theta_2 \\ = \frac{16k^3 r_1 p_{\theta,1}}{(12\wp - k)^3} + \frac{16k^3 p_{\theta,1}\dot{r}_1}{(12\wp - k)^3} - \frac{192k^3\dot{\wp}}{(12\wp - k)^4} r_1 p_{\theta,1} - \frac{4\theta_1 k^2 r_1}{(12\wp - k)^2}\end{aligned} \tag{27}$$

where $(r_1, \theta_1, p_{r,1}, p_{\theta,1})$ refer to solutions of the first variational system and $(r_2, \theta_2, p_{r,2}, p_{\theta,2})$ refer to solutions of the second variational system that we want to solve. The equations are now inhomogeneous, with left-hand sides corresponding to the (homogeneous) first variational equations, and right-hand sides formed of solutions of the first variational equations (which induces coupling).

These equations look nonlinear, at first. However, as explained in [17], the right-hand sides are formed of linear combinations of solutions of the second symmetric powers of the first variational system. Hence, the second variational system, together with the first, still reduces to a linear differential system and it makes sense to study its differential Galois group and its integrability. This fact remains true for variational equations of an arbitrary order.

As the first variational equations are solvable, we could write explicit solutions and then solve the second variational equations by variation of constants, but a better strategy is to proceed as in [38]: as the first variational equations are Lamé equations with Lamé–Hermite solutions, they have Abelian Galois group if and only if their formal solutions at zero do not

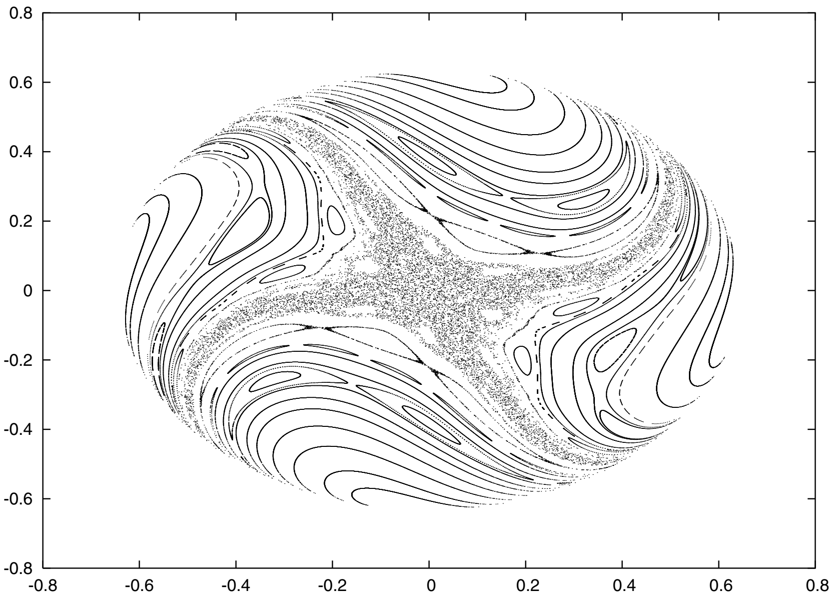

Figure 1. Poincaré cross section for the generalized spring-pendulum when $k = -a = 4/3$ and $E = -0.8$. The cross section plane with coordinates (θ, p_θ) is fixed at $r = 1$.

contain logarithms [17], and it is shown in [38] that this remains true for variational equations of arbitrary order.

This is easily tested in the following way: first we compute formal solutions (as a power series) of the first variational equations around zero. Then we plug a generic linear combination of these power series in the right-hand sides of (26) and (27). Next we apply the method of variation of constants: we thus have to integrate a (known) combination of power series and there is a logarithm if and only if this combination of power series has a non-zero residue (i.e. a term of degree -1 in its (Laurent) expansion in powers of t).

Performing this computation we show that the second variational equations are integrable. Iterating the process, we computed the solutions of the third, fourth, ..., until the seventh variational equations and found that they are all integrable. We could not continue the calculations to higher variational equations for the following reasons.

The first fact is that the size of the right-hand sides of the successive variational equations grows rapidly.

The second fact is that the valuation of the solutions decreases as the order of the variational equation grows. For example, the valuation of r_2 is -4, the valuation of r_3 is $-5, \ldots,$ and the valuation of r_7 is -9. To obtain r_6 with an accuracy up to the term of degree 0 (to obtain the terms of negative valuation properly, which is all we need for integrability by the above remarks), we need to start from an r_1 with 27 terms. To obtain r_7, we need to start from an r_1 with 30 terms, and so on. The combination of these two facts makes the computation intractable for the variational equations of order eight.

The fact that the valuations decrease is no surprise. We know that the restriction of the Hamiltonian flow to the invariant manifold $\mathcal{N}$ of section 3 is an integrable system with one degree of freedom. Calculations show that the corresponding solution $(r, 0, p_r, 0)$ has a valuation at zero that decreases just like the r_i above (and indeed seems to govern the lowest valuation in the r_i).

Now the fact that the variational equations up to order 7 are integrable might lead to a suspicion that the system could be integrable. However, numerical experiments clearly indicate chaotic behaviour which contradicts meromorphic integrability. We show an example of our numerical experiments in figure 1. In this figure we show the Poincaré cross section for energy $E = -0.8$ and $k = -a = 4/3$. On the level $H = E$ we chose (r, θ, p_θ) as coordinates. The cross-section plane was fixed at $r = 1$.

The model of the generalized swinging pendulum for $a = -k$ is thus a puzzling example of a system that seems (numerically) to be non-integrable but where even a deep application of the Morales-Ruiz–Ramis theory is not enough to detect this non-integrability rigorously. There is only one reported result concerning application of higher variational equations for proving non-integrability. In [38] Morales-Ruiz reports that for a certain case of the Henon–Heiles system the identity component of the differential Galois group of first- and second-order VEs is Abelian but for the third-order VE it is not. We also have several examples of Hamiltonian systems for which VE_k have Abelian identity component of differential Galois group for $k < 3$ but non-Abelian for $k = 3$. Thus, as far as we know the generalized spring-pendulum system with $a = -k$ is the only example where the application of higher order variational equations is unsuccessful in proving non-integrability.

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